Problems on General Relativity: 5

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Problem 1. Find the torsion free, metric connection ${\Gamma^i}_j$ and it's curvature two form ${R^i}_j$ given a cotangent frame (e^1, e^2, e^3) such that:

$$de^1 = e^2 \wedge e^3, \quad de^2 = e^3 \wedge e^1 \quad de^3 = -e^1 \wedge e^2$$

and a metric tensor:

$$g = e^1 \otimes e^1 + e^2 \otimes e^2 - e^3 \otimes e^3.$$

Helpful tip: Given a cotangent frame $(e^1, ..., e^n)$ such that

$$de^i = \frac{1}{2}{f^i}_{jk} e^j \wedge e^k, \qquad {f^i}_{jk} = -{f^i}_{kj}$$

and a metric tensor

$$g = g_{ij} e^i \otimes e^j, \qquad g_{ij} = g_{ji} = \text{const}$$

the corresponding torsion free and metric connection is given by

$${\Gamma^i}_{jk} = \frac{1}{2} g^{il}(f_{ljk} + f_{jkl} - f_{klj}).$$

The curvature 2-form always is

$${R^i}_j = d{\Gamma^i}_j + {\Gamma^i}_k \wedge {\Gamma^k}_j$$

Problem 2. Given $(e^0, e^1, e^2, e^3) = (dt, dx, dy, dz)$, and a metric tensor

$$g = -dt \otimes dt + a^2(t)(dx \otimes dx + dy \otimes dy + dz \otimes dz)$$

calculate the torsion free metric connection ${\Gamma^a}_b$, and its curvature ${R^a}_b$ 2-form.
Helpful tip:
Given $(e^1, ..., e^n) = (dx^1, ..., dx^n)$, and a metric tensor

$$g = g_{ab} dx^a \otimes dx^b, \qquad g_{ab} = g_{ba},$$

the corresponding torsion free metric connection ${\Gamma^a}_b$ is given by

$${\Gamma^a}_{bc} = \frac{1}{2} g^{ad}(g_{db,c} + g_{dc,b} - g_{bc,d})$$

Problem 3* - optional.

$$de^1 = e^2 \wedge e^3, \quad de^2 = e^3 \wedge e^1 \quad de^3 = \pm e^1 \wedge e^2$$

$$g = e^1 \otimes e^1 + e^2 \otimes e^2 \pm e^3 \otimes e^3.$$

Show, that

$$\mathcal{L}_{e_1} g = \mathcal{L}_{e_2} g = \mathcal{L}_{e_3} g = 0$$

Hint:

$$\mathcal{L}_X \omega = X \lrcorner d\omega + d(X \lrcorner \omega)$$